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Phase transitions in the random field Ising model in the presence of a transverse field

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Abstract. We have studied the phase transition behaviour of the random field Ising model in the presence of a transverse (or tunnelling) field. The mean field phase diagram has been studied in detail, and in particular the nature of the transition induced by the tunnelling (transverse) field at zero temperature. A modified hyperscaling relation for the zero-temperature transition has been derived using the Suzuki–Trotter formalism and a modified 'Harris criterion'. A mapping of the model to a randomly diluted antiferromagnetic Ising model in uniform longitudinal and transverse field is also given.

1. Introduction

The random field Ising model (RFIM), described by the Hamiltonian

$$H = -\sum_{ij} J_{ij} S_i^z S_j^z - \sum_i h_i S_i^z$$

where $S_i = \pm 1$ are the Ising spins and the h_i are independent quenched random variables with mean zero, has been subjected to rigorous theoretical and experimental investigations in recent years [1]. The random field acts as an order-destroying field, which effectively reduces the transition temperature T_c of the classical Ising transition from the symmetry broken (ferromagnetic) phase to the symmetric phase (configuration-averaged magnetization zero) as the magnitude of the random field is of the random field h_r^c (i.e. there exists a critical line $h_r(T)$ in the $h_r{-}T$ diagram). For $h_r > h_r^c$, the system is always disordered at any temperature. It has been established [2] that RFIM does not order for $d \leqslant 2$, indicating the lower critical dimensionality for the system to be two. The existence of long-range order in the three-dimensional model, for low temperature and weak random field, has been rigorously proved [2]. It has also been established from the mean field studies of the classical model that whenever the distribution function of the random field $P(h)$ has a minimum at zero field (e.g. the binary distribution) one obtains a tricritical point [3] on the critical line, so that the transition for the larger values of the random field is discontinuous, whereas if the distribution function $P(h)$ decreases monotonically with the increase of the magnitude of h (e.g. the Gaussian distribution), the transition is always continuous [4]. One should also mention at this point that the existence of a tricritical point has not been observed for any finite-dimensional systems. If the transition is second order, the scaling arguments [5–7] (based on the assumptions that near the critical point $T_c(h_r)$ the random field fluctuations dominate over the thermal fluctuations), suggest a modified hyperscaling relation of the form $2-\alpha = \nu(d-\theta)$, with the exponents ν and α as the correlation length and specific heat exponents, respectively. The new exponent θ is related to the exponents η and $\bar{\eta}$ (where η and $\bar{\eta}$ describe the decay of the connected and disconnected correlation functions, respectively, at $T_c(h_r)$) through the relation $\theta = 2 + \eta - \bar{\eta}$. Obviously there seem to exist three independent critical exponents, but recent accurate high-temperature series expansion studies [8] imply that $\theta = 2 - \eta$, and $\bar{\eta} = 2\eta$ so that the Schwartz–Soffer inequality [9] is fulfilled as an equality. One should also mention that the static universal critical behaviour is expected to be the same for ferromagnets in a random field and dilute antiferromagnets in a uniform field [10].

The study of the classical Ising systems in the presence of tunable quantum fluctuations, namely the transverse field, dates back to the early 1960s. In recent years numerous efforts have been made to study the short-range (Edwards–Anderson type) as well as the long-range (Sherrington–Kirkpatrick type) Ising spin glass in the presence of a transverse (tunnelling) field, using approximate analytical and various numerical (e.g. quantum Monte Carlo) techniques. These investigations explore the effect of quantum fluctuations (due to quantum tunnelling) on the classically frustrated states [11].

It has been conjectured that frustration in the RFIM gives rise to a 'many-valley' structure in the configuration space, similar to the situation in spin glasses [1]. We therefore study the random (longitudinal) field transverse Ising model (RFTIM), to analyse the effects of quantum fluctuations (induced by the transverse or tunnelling field) on the transition in the RFIM. Specifically, we consider an RFTIM system represented by the Hamiltonian

$$H = -\sum_{ij} J_{ij} S_i^z S_j^z - \sum_i h_i S_i^z - \Gamma \sum_i S_i^x. \tag{1}$$

In view of the fact that the zero-temperature transverse field driven transition in the transverse Ising models (TIM) may be equivalent to the thermal phase transition in a higher-dimensional classical Ising system [12]), and employing the hyperscaling relation $2 - \alpha = \nu(d - \theta)$, one can also obtain a modified hyperscaling relation for the zero-temperature transition in an RFTIM system.

2. Mean field studies

We consider a random field Ising ferromagnet (with long-range interaction), in the presence of a uniform transverse field

$$H = -\frac{J}{N} \sum_{i \neq j} S_i^z S_j^z - \sum_i h_i S_i^z - \Gamma \sum_i S_i^x \tag{2}$$

where S_i^α are Pauli spin operators satisfying the commutation relations $[S_i^\alpha, S_j^\beta] = \mathrm{i}\epsilon_{\alpha\beta\gamma}\delta_{ij} S_i^\gamma$, Γ is the strength of the tunnelling field and h_i, as mentioned earlier, is the quenched random field at each site with a probability distribution $P(h)$, having zero mean and nonzero variance.

Using the replica trick and the saddle-point integration (in the $N \to \infty$ limit), following [4], the above quantum Hamiltonian can be reduced to an effective single site Hamiltonian given by (see appendix A)

$$H = -\sum_i (2m^z J + h_i) S_i^z - \Gamma \sum_i S_i^x \tag{3}$$

where m^z is the configuration-averaged longitudinal magnetization. The configuration-averaged magnetization vector can be readily written [13] in the self-consistent form

$$\boldsymbol{m} = \overline{\tanh \beta \left[\sqrt{(2m^z J + h)^2 + \Gamma^2}\right] \left(\frac{(2m^z J + h)\hat{z} + \Gamma \hat{x}}{\sqrt{(2m^z J + h)^2 + \Gamma^2}}\right)} \tag{4}$$

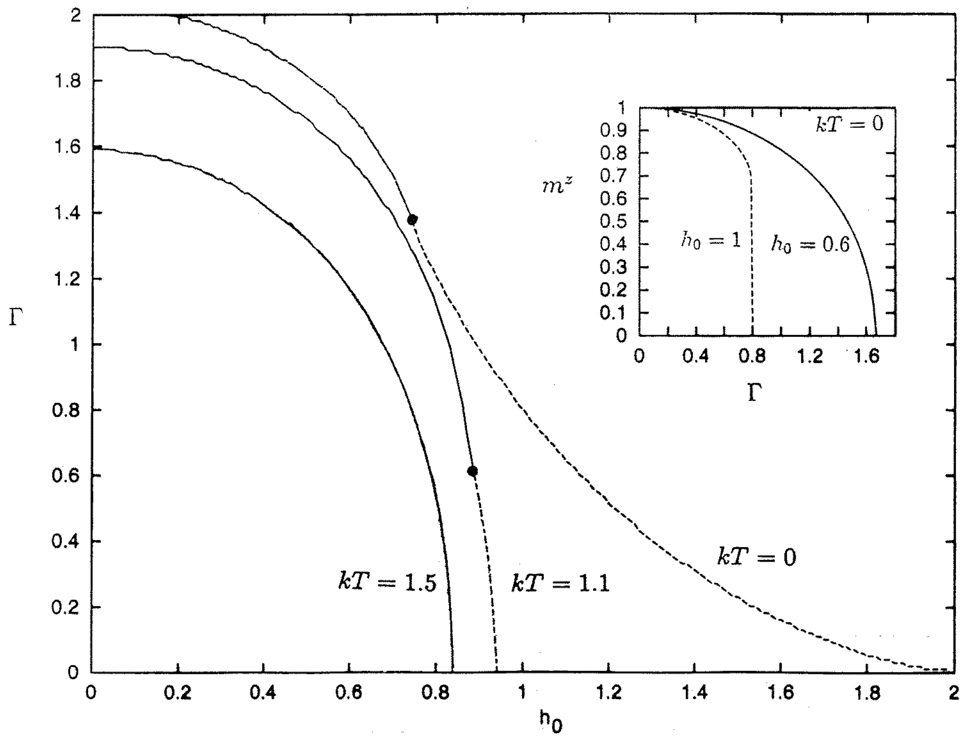

Figure 1. The mean field phase diagram of the RFTIM in the Γ–h_0 plane for different temperatures. The black circle denotes the tricritical point. The inset shows the nature of the transition below and above the tricritical point.

so that the configuration-averaged longitudinal magnetization is

$$m^z = \overline{\left[\tanh\beta(\sqrt{(2m^zJ+h)^2+\Gamma^2})\frac{2m^zJ+h}{\sqrt{(2m^zJ+h)^2+\Gamma^2}}\right]} \tag{5}$$

where the overhead bar denotes a configuration average over the distribution of the random field. If one now uses a binary distribution of the random field

$$P(h) = \tfrac{1}{2}\delta(h-h_0) + \tfrac{1}{2}\delta(h+h_0) \tag{6}$$

the configuration-averaged longitudinal magnetization can be written as [14]

$$m^z = \frac{1}{2}\left[\tanh\beta\left(\sqrt{(2m^zJ+h_0)^2+\Gamma^2}\right)\frac{2m^zJ+h_0}{\sqrt{(2m^zJ+h_0)^2+\Gamma^2}}\right] + \frac{1}{2}\left[\tanh\beta\left(\sqrt{(2m^zJ-h_0)^2+\Gamma^2}\right)\frac{2m^zJ-h_0}{\sqrt{(2m^zJ-h_0)^2+\Gamma^2}}\right]. \tag{7}$$

From equation (5), one can conclude that for any symmetric distribution $P(h)$ of the random field, $m^z = 0$ is always a solution of (5). For large enough temperature and random field, this is the only solution. At low temperature and weak random field, one finds an additional solution $m^z \neq 0$ (symmetry-broken phase) with lower free energy. If the transition is continuous, one can find the transition point by expanding (5) around $m^z = 0$:

$$m^z \sim am^z - b(m^z)^3 - c(m^z)^5 - \cdots. \tag{8}$$

A second-order transition is found when $a = 1$ as long as $b > 0$. If $b < 0$ the transition is first order and the point $a = 1$ and $b = 0$ characterizes a tricritical point on the phase boundary, separating the ferromagnetic phase ($m^z \neq 0$) from the phase with $m^z = 0$ (but with nonzero value of the configuration-averaged squared magnetization). In the classical case ($\Gamma = 0$) [3], one finds $a = 2\beta J\overline{(1 - t^2)}$; $b = \frac{1}{3}(2\beta J)^3\overline{[(1 - t^2)(1 - 3t^2)]}$, where $t = \tanh \beta h$. With a binary distribution of the random field one finds the tricritical point [3] at $\beta J = \frac{3}{4}$, $\tanh^2(\beta h_0) = \frac{1}{3}$. One can solve equation (6) (with $\Gamma = 0$) numerically, to obtain the entire phase diagram of the classical system.

In the extreme quantum limit ($T = 0$), the thermal fluctuations are absent and the fluctuations induced by the random field and quantum fluctuations due to the transverse field tend to destroy the long-range order. From (5) the configuration-averaged longitudinal magnetization can be written as

$$m^z = \overline{\left[\frac{2m^z J + h}{\sqrt{(2m^z J + h)^2 + \Gamma^2}}\right]}. \tag{9}$$

Expanding the magnetization in the form (8), we find for any symmetric distribution of the random field

$$a = \overline{\left[\frac{2J}{\sqrt{h^2 + \Gamma^2}} - \frac{2Jh^2}{(h^2 + \Gamma^2)^{3/2}}\right]} \tag{10a}$$

$$b = \overline{\left[\frac{24J^3}{(h^2 + \Gamma^2)^{3/2}} - \frac{144h^2 J^3}{(h^2 + \Gamma^2)^{5/2}} + \frac{120J^3 h^4}{(h^2 + \Gamma^2)^{7/2}}\right]}. \tag{10b}$$

Specifically, if we use the binary distribution of the random field (6),

$$a = \left[\frac{2J}{\sqrt{h_0^2 + \Gamma^2}} - \frac{2Jh_0^2}{(h_0^2 + \Gamma^2)^{3/2}}\right] \tag{11a}$$

$$b = \left[\frac{24J^3}{(h_0^2 + \Gamma^2)^{3/2}} - \frac{144h_0^2 J^3}{(h_0^2 + \Gamma^2)^{5/2}} + \frac{120h_0^4 J^3}{(h_0^2 + \Gamma^2)^{7/2}}\right]. \tag{11b}$$

The tricritical point ($a = 1, b = 0$) is obtained at $\Gamma \cong 1.4J$, $h_0 \cong 0.74J$. The numerically obtained phase diagram is very similar to the phase diagram obtained in the classical case ($\Gamma = 0$), indicating that the transverse field behaves in the same manner as the temperature to destroy the long-range order.

When both thermal and quantum fluctuations are present, we obtain the phase diagram in the Γ–h_0 plane (for various temperatures below the pure system transition temperature) by numerically solving (6), and also if the transition is second order the transition point is given by

$$a = \overline{\left[\frac{4h^2 J\beta(1 - t^2)}{2(h^2 + \Gamma^2)} + \frac{2tJ}{(h^2 + \Gamma^2)^{1/2}} - \frac{2th^2 J}{(h^2 + \Gamma^2)^{3/2}}\right]} = 1 \tag{12}$$

where $t = \tanh \beta h$. We find, from the numerically obtained phase diagram, that as the temperature is increased, the phase diagram shrinks to lower values of Γ and h_0 and the tricritical point on the critical line in the Γ–h_0 plane shifts to a higher value of h_0 (i.e. the second-order region on the phase boundary increases) and eventually if the temperature is higher than the value at the tricritical point of the classical phase boundary, the entire phase boundary corresponds to the continuous transition. These mean field calculations can be readily extended to obtain numerically the phase diagram when the random field distribution is Gaussian with zero mean and nonzero variance

$$P(h) = \frac{1}{\sqrt{2\pi\Delta^2}} \exp\left(-\frac{h^2}{2\Delta^2}\right). \tag{13}$$

One can easily see in this case that the phase transition, obtained from mean field studies, is continuous for all values of Γ and Δ (width of the Gaussian distribution) because even in the limit of temperature and transverse field both being zero, the transition driven by the random field is continuous [3].

3. Zero temperature scaling behaviour

Let us consider a (nearest-neighbour, ferromagnetic) RFIM in the presence of a transverse field in d dimensions

$$H = -J \sum_{\langle ij \rangle} S_i^z S_j^z - \sum_i h_i S_i^z - \Gamma \sum_i S_i^x. \tag{14}$$

with the distribution of the random field being Gaussian (9). The zero-temperature transition in a transverse Ising model on a d-dimensional lattice is equivalent to the finite-temperature thermal phase transition in an extremely anisotropic classical Ising Hamiltonian with one added dimension, namely the Trotter dimension. Using the Suzuki–Trotter formalism [15], we obtain the equivalent classical Hamiltonian for the RFIM in a transverse field

$$\begin{aligned} H_{eff} = -(J/M) \sum_{\langle ij \rangle} \sum_{k=1}^{M} S_{ik} S_{jk} - \sum_i \sum_{k=1}^{M} \frac{h_i}{M} S_{ik} \\ -(1/2\beta) \log \coth(\beta\Gamma/M) \sum_i \sum_k S_{ik} S_{j,k+1} \end{aligned} \tag{15}$$

where k indicates the Trotter direction. In the zero-temperature limit ($M \to \infty$), the quantum transition in the original quantum Hamiltonian (14) falls in the same universality class with the thermal phase transition in the equivalent classical anistropic RFIM (15) with the disorder (random field) correlated (striped) in the Trotter direction. To see whether in this classical anisotropic higher-dimensional Ising system with striped randomness, the fluctuations induced by the random field dominate over the thermal fluctuations, we consider the 'Harris criterion' [16] for the systems with randomness correlated in one particular (Trotter) direction. If we consider a domain having the dimension of the order of the correlation length ξ, the fluctuation in the critical temperature due to randomness is given as [17]

$$\Delta T_c \sim \xi^{-d/2} \sim (\Delta T)^{\mathrm{d}\nu/2} \tag{16}$$

since the randomness is correlated in a particular (Trotter) direction and the correlation length for the pure system diverges as $(T - T_c)^{-\nu}$. So the random field is now a relevant parameter if $\mathrm{d}\nu < 2$, or

$$\alpha + \nu > 0 \tag{17}$$

where both α and ν are exponents for the pure classical Ising system in $(d+1)$ dimensions. Since (17) is satisfied for two-dimensional ($\alpha = 0,\ \nu = 1$) as well as any higher-dimensional pure classical Ising systems one expects that in the equivalent classical RFIM with correlated randomness, the fluctuations induced by the random fields are relevant and dominate over the thermal fluctuations.

Since, in the zero-temperature transition in RFTIM, the disorder-induced fluctuations seem to dominate, we assume that, like the singular part of free energy in the finite-temperature transition in the classical RFIM, here also, the singular part of the ground state energy scales as $E_{sing} \sim \xi^{\theta}$. The singular part of the ground state energy density should then scale as

$$E_{sing}/(\text{correlation volume}) = E_{sing}/(\xi^d \xi_\tau)$$

where $\xi^d \xi_\tau$ is the correlation volume in the quantum phase transition and $\xi_\tau = \xi^z$ (z is the dynamical exponent of the quantum system) is the correlation length in the imaginary time direction [18]. If one assumes the ground state energy to grow with the critical interval $[(\Delta/J)^* - (\Delta/J)]$, away from the random field fixed point $(\Delta/J)^*$, as $[(\Delta/J)^* - (\Delta/J)]^{2-\alpha}$, and the correlation length has a divergence of the form $[(\Delta/J)^* - \Delta/J)]^{-\nu}$, the ground state energy density should then scale as

$$E_{sing}\xi^{\theta-d-z} \sim [(\Delta/J)^* - \Delta/J]^{\nu(d+z-\theta)} \sim [(\Delta/J)^* - \Delta/J]^{2-\alpha}.$$

From the above scaling relation for the RFTIM systems, we now expect a modified hyperscaling relation for the zero-temperature transition in the RFTIM system, of the form $2-\alpha = (d+z-\theta)\nu$. If one now accepts the relation $\theta = 2+\eta-\bar{\eta}$ along with $\bar{\eta} = 2\eta$, the modified hyperscaling relation for the RFIM system is written as $2-\alpha = (d+z+\eta-2)$.

We must conclude this section with a note of caution. The above scaling form is really a result of the application of the modified Harris criterion to a classical system (with disorder correlated in one particular direction) which is obtained using the Suzuki–Trotter formalism to the original quantum Hamiltonian. It essentially relies on the assumption that in the case of classical RFIM the random field fluctuations dominate over the thermal fluctuations, which leads to the dimensional reduction [5–7]. Recent numerical studies of three-dimensional classical random field Ising models using both binary and Gaussian distribution of the random field [19] and also experimental studies (see Rieger [11, 19]), however, indicate a violation of the hyperscaling relation $2-\alpha = \nu(d-\theta)$. These studies support the two-exponent scaling scenario. However, the value of the exponent β is consistent with zero, which indicates a discontinuous jump in the order parameter at the transition temperature though no latent heat and no divergence for the specific heat is found. Also, the possibility of a spin glass phase intermediate between the para and the ferro phase has also been discussed [20]. In view of these theoretical uncertainties for the classical case, the hyperscaling relation obtained here might require a closer scrutiny.

4. Mapping of the random Ising antiferromagnet in uniform longitudinal and transverse field to the RFTIM

As mentioned earlier, Fishman and Aharony and independently Cardy [10] showed the equivalence of the transition in the RFIM to the transitions in a dilute Ising antiferromagnet in a steady field. Here we study the problem of equivalence in the presence of a noncommuting transverse field and show that the random Ising antiferromagnet in a uniform transverse and longitudinal field (RIAFTL) is expected to be in the universality class of the Ising ferromagnet with uniform transverse field and random longitudinal field (RFTIM). This equivalence is obtained, in a semiclassical approximation neglecting commutations, via a decimation of one sublattice of the RIAFTL system. We illustrate the procedure by considering first the one-dimensional model, commenting later on generalizations.

The decimation procedure is a partial trace over sites of one sublattice, e.g. that in which the site label i is odd. To rearrange the statistical weights of the remaining spins, the original (reduced) Hamiltonian

$$-\beta H = \sum_i (-K_{i,i+1} S_i^z S_{i+1}^z + h_i S_i^z + \Gamma S_i^x) = \sum_i H_i \tag{18}$$

will be mapped into a new form

$$-\beta H' = \sum_i (-K'_{2i,2i+2} S_{2i}^z S_{2i+2}^z + h'_{2i} S_{2i}^z + \Gamma' S_{2i}^x) = \sum_i H'_{2i}. \tag{19}$$

In (1), h and Γ are longitudinal and components of a uniform field, and the label i on h_i is there only to allow for the effects of site dilution (h_i is independent of i in the case of bond dilution). $K_{i,i+1}$ is a random antiferromagnetic exchange. The semiclassical decimation procedure, which neglects commutations but is otherwise exact, is as follows:

$$\begin{aligned}\prod_i [\mathrm{Tr}_{S_{2i+1}}[\exp(H_i)]] &= \exp(h_{2i} S_{2i}^z + \Gamma_{2i} S_{2i}^x) \\ &\quad \times \mathrm{Tr}_{S_{2i+1}} \exp(S_{2i+1}^z [h_{2i+1} - K_{2i,2i+1} S_{2i}^z - K_{2i+1,2i+2} S_{2i+2}^z] + \Gamma S_{2i+1}^x) \\ &\quad \times \exp(h_{2i+2} S_{2i+2}^z + \Gamma_{2i+2}^x) \cdots \\ &= \text{constant} \prod_i \exp(H'_{2i}). \end{aligned} \tag{20}$$

The trace over S_{2i+1} produces the factor

$$b(S_{2i}^z, S_{2i+2}^z) = 2\cosh[(h_{2i+1} - K_{2i,2i+1} S_{2i}^z - K_{2i+1,2i+2} S_{2i+2}^z)^2 + \Gamma^2]^{1/2}. \tag{21}$$

This can be written as

$$\exp(A + B S_{2i}^z + C S_{2i+2}^z + D S_{2i}^z S_{2i+2}^z)$$

where matching of the expression for all four possible sets of values for (S_{2i}^z, S_{2i+2}^z) gives A, B, C, D in terms of Γ, h_{2i+1}, $K_{2i,2i+1}$, ,$K_{2i+1,2i+2}$ (see appendix B for details). We thus arrive at the recursion relations

$$h'_{2i} = h_{2i} + B(K_{2i,2i+1}, K_{2i+1,2i+2}) + C(K_{2i-2,2i-1}, K_{2i-1,2i}) \tag{22}$$

$$\Gamma' = \Gamma \tag{23}$$

$$K'_{2i,2i+2} = D(K_{2i,2i+1}, K_{2i+1,2i+2}). \tag{24}$$

The particular case $h_i = 0$ of this shows that the random bond Ising antiferromagnet in a uniform transverse field maps to a random bond Ising ferromagnet in a uniform transverse field.

The general case (h_i, Γ both nonzero) maps to a random longitudinal field model, along with uniform transverse field. This is most easily illustrated for the random bond case when h and Γ are both independent of the site label i. For $h \ll K$, one can simplify $B(K_1, K_2)$ (where K_1 and K_2 are two neighbouring bonds) (see appendix B):

$$B(K_1, K_2) = -\frac{h}{2}\left[\frac{\Lambda_+}{\Omega_+}\tanh\Omega_+ + \frac{\Lambda_-}{\Omega_-}\tanh\Omega_-\right] + \mathrm{O}(h^2) \tag{25}$$

with

$$\Omega_\pm = [\Lambda_\pm^2 + \Gamma^2]^{1/2} \qquad \Lambda_\pm = K_1 \pm K_2. \tag{26}$$

For the case of bond dilution, where

$$K_{i,i+1}\begin{cases} = K & \text{with probability } p \\ = 0 & \text{with probability } (1-p) \end{cases}$$

it is clear that Λ_+ is always positive, while Λ_- could be positive or negative with equal probability for any nonzero value of the probability p. The result is that h'_{2i} is distributed in such a way that its mean is not zero, but it divides into two parts and the part (containing Λ_-) which couples to the critical fluctuations (antiferromagnetic, in the original model) has zero mean, whereas the part with nonzero mean couples to the the ferromagnetic order parameter. K', on the other hand, is a random (ferromagnetic) exchange. This indicates that the model is expected to have the universality class of the RFTIM model at least in the limit of small field h. The same procedure can be extended in higher dimensions using cluster approximation of the type common in decimation methods [17], again with the same conclusion. Equations (21)–(25) give the expected relationship between the parameters of the original system and the resulting RFTIM.

5. Summary and concluding remarks

We have studied the phase transition behaviour of the random field Ising model in the presence of a transverse field. This transverse field represents the (quantum) tunnelling fluctuations in double-well systems representing the model order–disorder ferroelectric systems, Jahn–Teller systems, etc [13]. The mean field phase diagram has been studied in detail, in particular at zero temperature, where the transition is governed by the fluctuations induced by the random field and quantum fluctuations due to the transverse field. An effective hyperscaling relation has been derived for the zero-temperature (quantum) transition in the RFTIM system. This scaling is based on the use of the hyperscaling relation for the thermal phase transition in the equivalent classical system and the application of the Suzuki–Trotter formalism to the original quantum Hamiltonian. Also the dynamical exponent of the quantum model which appears in determining the correlation volume in the quantum phase transition has been included in the hyperscaling relation. However, these arguments are not sufficient to prove if the RFTIM has any phase transition (at any nonvanishing tunnelling field Γ_c) in one dimension. Also, as discussed earlier, the present theoretical uncertainties in the classical RFIM indicate an eventual requirement of some modification of these scaling results.

It has also been shown by a semiclassical procedure that the ferromagnetic transverse Ising model with a random longitudinal field provides the universal critical behaviour of the random (e.g. randomly diluted) Ising antiferromagnet in a uniform field having both transverse and longitudinal components. This is shown by employing a sublattice decimation on the random antiferromagnet in a general uniform field. Although the decimation procedure is only demonstrated for a one-dimensional system, it can be generalized for the higher dimensions. This mapping also indicates the possible application of the results of the studies for RFTIM to random quantum (Ising) antiferromagnets.

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Appendix A. Effective single-site Hamiltonian for a long-range interacting RFTIM

To derive the effective single-site Hamiltonian we consider the Hamiltonian of a (long-range ferromagnetic) RFIM in a transverse field

$$H = -\frac{J}{N}\sum_{i\neq j} S_i^z S_j^z - \sum_i h_i S_i^z - \Gamma \sum_i S_i^x \tag{A1}$$

where the random variable at each site satisfies a Gaussian distribution (10). The configuration-averaged free energy of the system is given by

$$F = -kT\overline{\log Z} \tag{A2}$$

where k is the Boltzmann constant and Z is the partition function for a particular realization of the random fields. Using a replica trick [4] we can write the n-replicated free energy in the form

$$F = -kT \lim_{n\to 0} \frac{1}{n}\overline{(Z^n - 1)} \tag{A3}$$

$$= -kT \lim_{n\to 0}\Bigg(\frac{1}{n}\Bigg[\overline{\mathrm{Tr}\exp\Bigg(-\beta\sum_{\alpha=1}^{n} H_0(\alpha)\Bigg)}$$
$$\overline{\times P\exp\Bigg(\int_0^\beta \mathrm{d}\tau \sum_{\alpha=1}^{n}\sum_{ij}\frac{J}{N} S_{\alpha i}^z(\tau) S_{\alpha j}^z(\tau) + \sum_i h_i S_{\alpha i}^z(\tau)\Bigg)}\Bigg] - 1\Bigg) \tag{A4}$$

where α denotes the αth replica, P denotes the time ordering, $H_0(\alpha) = -\Gamma\sum_i S_{\alpha i}^x$ and the $S^z(\tau)$'s are operators in the interaction representation. We can now perform the configuration averaging to obtain

$$= -kT \lim_{n\to 0}\Bigg(\frac{1}{n}\Bigg[\mathrm{Tr}\exp\Bigg(-\beta\sum_{\alpha=1}^{n} H_0(\alpha)\Bigg)$$
$$\times P\exp\Bigg(\int_0^\beta \mathrm{d}\tau \sum_{\alpha=1}^{n}\sum_{ij}\frac{J}{N} S_{\alpha i}^z(\tau) S_{\alpha j}^z(\tau)$$
$$+\frac{\Delta^2}{2}\sum_i\Bigg(\sum_{\alpha=1}^{n}\int_0^\beta \mathrm{d}\tau\, S_{\alpha i}^z(\tau)\Bigg)^2\Bigg)\Bigg] - 1\Bigg). \tag{A5}$$

A Hubbard–Stratonovitch transformation simplifies the term

$$\exp\Bigg[\int_0^\beta \mathrm{d}\tau \sum_{\alpha}\sum_{ij}\frac{J}{N} S_{\alpha i}^z(\tau) S_{\alpha j}^z(\tau)\Bigg] = \exp\Bigg[\int_0^\beta \mathrm{d}\tau \sum_{\alpha=1}^{n}\Bigg|\sqrt{\frac{J}{N}}\sum_{i=1}^{N} S_{\alpha i}^z(\tau)\Bigg|^2\Bigg] \tag{A6}$$

(where the terms of order $(1/N)$ are neglected), so that we obtain the configuration-averaged n-replicated free energy

$$F = -kT \lim_{n\to 0}\frac{1}{n}\int_{-\infty}^{\infty}\prod_{\alpha=1}^{n}\mathrm{d}x_\alpha \left(\frac{N}{2\pi}\right)^{\frac{1}{2}}\Bigg(\mathrm{Tr}\exp\Bigg(N\beta\sum_{\alpha=1}^{n} S_\alpha^x\Bigg)$$
$$\times P\exp N\Bigg(-\frac{1}{2}\beta\sum_{\alpha=1}^{n} x_\alpha^2 + \sqrt{2J}\sum_{\alpha=1}^{n} x_\alpha \int_0^\beta S_\alpha^z(\tau) + \frac{\Delta^2}{2}\Bigg(\int_0^\beta S_\alpha^z(\tau)\Bigg)^2\Bigg)\Bigg) - 1 \tag{A7}$$

where the x_α's are dummy variables. In the $N \to \infty$ limit, one can readily obtain the saddle point configuration-averaged free energy

$$F = -kT \lim_{n\to 0} \frac{1}{n}\left[-\tfrac{1}{2}\beta \sum_{\alpha=1}^{n} x_\alpha^2 + \log \mathrm{Tr}\exp(A)\right] \tag{A8}$$

where

$$\exp(A) = \exp\left(\beta\Gamma \sum_{\alpha=1}^{n} S_\alpha^x P \exp\left(\sqrt{2J}\sum_{\alpha=1}^{n} x_\alpha \int_0^\beta S_\alpha^z(\tau) + \frac{\Delta^2}{2}\left(\sum_{\alpha=1}^{n}\int_0^\beta \mathrm{d}\tau\, S_\alpha^z(\tau)\right)^2\right)\right). \tag{A9}$$

The square term appearing in the above expression can be simplified using once again a Hubbard–Stratonovitch transformation to obtain

$$\exp(A) = \int_{-\infty}^{\infty} \frac{\mathrm{d}s}{(2\pi)^{\frac{1}{2}}} \exp\left(-\frac{s^2}{2}\right)\exp\left(\beta\Gamma\sum_{\alpha=1}^{n} S_\alpha^x\right)$$
$$\times P\exp\left(\sqrt{2J}\sum_{\alpha=1}^{n} x_\alpha \int_0^\beta S_\alpha^z(\tau)\,\mathrm{d}\tau + s\Delta \int_0^\beta \mathrm{d}\tau \sum_{\alpha=1}^{n} S_\alpha^z\right) \tag{A10}$$

where s is a dummy variable. Finally, one obtains the form of free energy (with $x = m^z\sqrt{2J}$ and $s\delta = h$) given by

$$F = -kT\left[-J(m^z)^2\beta + \int_{-\infty}^{\infty} \frac{\mathrm{d}h}{\sqrt{2\pi\Delta^2}} \exp\left(-\frac{h^2}{2\Delta^2}\right) \log\ \mathrm{Tr}\exp(\beta\Gamma S^x)\right.$$
$$\left.\times P\exp\left((2m^z J + h)\int_0^\beta S^z(\tau)\right)\right]$$
$$= -kT\left[-J(m^z)^2\beta + \int_{-\infty}^{\infty} \mathrm{d}h\, P(h) \log \mathrm{Tr}\exp(\beta(\Gamma S^x + (2m^z J + h)S^z))\right]. \tag{A11}$$

We have thus reduced the many-body Hamiltonian (in the $N \to \infty$ limit) to an effective single-site problem, where the molecular field at each site is given by $(2m^z J + h)$ where h is distributed with a probability distribution $P(h)$.

Appendix B. Mapping of a random Ising antiferromagnet in a uniform longitudinal and transverse field to the RFTIM

The equivalence between the transition in the RIAFTL system to that in the RFTIM system is obtained by employing semiclassical decimation of the one sublattice of the RIAFTL system, which neglects commutators between the spin operators. Here a partial trace is done over sites of one sublattice, e.g. that in which the site label i is odd. The original (reduced) Hamiltonian

$$-\beta H = \sum_i (-K_{i,i+1} S_i^z S_{i+1}^z + h_i S_i^z + \Gamma S_i^x) = \sum_i H_i \tag{B1}$$

is mapped into a new form

$$-\beta H' = \sum_i (-K'_{2i,2i+2} S_{2i}^z S_{2i+2}^z + h'_{2i} S_{2i}^z + \Gamma' S_i^x) = \sum_i H'_{2i}. \tag{B2}$$

The trace over S_{2i+1} produces the factors

$$b(S_{2i}^z, S_{2i+2}^z) = 2\cosh[(h_{2i+1} - K_{2i,2i+1} S_{2i}^z - K_{2i+1,2i+2} S_{2i+2}^z)^2 + \Gamma^2]^{1/2}. \tag{B3}$$

This can be written as

$$\exp(A + BS^z_{2i} + CS^z_{2i+2} + DS^z_{2i}S^z_{2i+2})$$

where matching of the expression for all four possible sets of values for (S^z_{2i}, S^z_{2i+2}) gives A, B, C, D in terms of Γ, h_{2i+1}, $K_{2i,2i+1}$, ,$K_{2i+1,2i+2}$. For example

$$B = \frac{1}{4}\log\left[\frac{b(1,1)b(1,-1)}{b(-1,-1)b(-1,1)}\right] \equiv B(K_{2i,2i+1}, K_{2i+1,2i+2}) = C(K_{2i+1,2i+2}, K_{2i,2i+1}) \quad \text{(B4)}$$

$$D = \frac{1}{4}\log\left[\frac{b(1,1)b(-1,-1)}{b(1,-1)b(-1,1)}\right] \equiv D(K_{2i,2i+1}, K_{2i+1,2i+2}) \quad \text{(B5)}$$

so that we arrive at the recursion relations (22), (23) and (24).

For $h \ll K$, one can evaluate $B(K_1, K_2)$ (where K_1 and K_2 are two neighbouring bonds), using the simplified relations

$$b(1,1) = 2\cosh\left[\Omega_+ - h\frac{\Lambda_+}{\Omega_+}\right]$$

$$b(1,-1) = 2\cosh\left[\Omega_- - h\frac{\Lambda_-}{\Omega_-}\right]$$

$$b(-1,-1) = 2\cosh\left[\Omega_+ + h\frac{\Lambda_+}{\Omega_+}\right]$$

$$b(-1,1) = 2\cosh\left[\Omega_+ + h\frac{\Lambda_-}{\Omega_-}\right]$$

where

$$\Omega_\pm = [\Lambda^2_\pm + \Gamma^2]^{1/2} \qquad \Lambda_\pm = K_1 \pm K_2. \quad \text{(B6)}$$

Hence

$$B(K_1, K_2) = \frac{1}{4}\log\left[\frac{\cosh(\Omega_+ - h\Lambda_+/\Omega_+)\cosh(\Omega_- - h\Lambda_-/\Omega_-)}{\cosh(\Omega + + h\Lambda_+/\Omega_+)\cosh(\Omega_- + h\Lambda_-/\Omega_-)}\right].$$

If we now use the relation (for small h)

$$\log\left[\frac{\cosh(\alpha + \gamma h)}{\cosh(\alpha - \gamma h)}\right] = 2\gamma h \tanh\alpha + \cdots$$

we get

$$B(K_1, K_2) = -\frac{h}{2}\left[\frac{\Lambda_+}{\Omega_+}\tanh\Omega_+ + \frac{\Lambda_-}{\Omega_-}\tanh\Omega_-\right] + \mathrm{O}(h^2) \quad \text{(B7)}$$

etc.

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